# Infinite Dimensional Topologies

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## 1 Topologies

Before we discuss what a topology is, it is instructive to remember a topology we're already somewhat familiar with.

### 1.1 Intervals of the Real Line

We begin with an example from $\mathbb{R}$. We define open and closed intervals of $\mathbb{R}$ as follows:

$$(a,b) := \{x : a < x < b\} \text{ we call this an open interval}$$

$$[a,b] := \{x : a \leq x \leq b\} \text{ we call this a closed interval}$$

note that the union and intersection of two overlapping open or closed intervals is again a interval of the same type. However, we can have a *infinite* intersection of intervals of one type which gives the other type of interval. Namely, consider the intersection of all open intervals containing the closed interval $[0,1]$:

$$\bigcap_{a<0,\, b>1} (a,b)$$

this gives us $[0,1]$ itself, which is closed. We are now ready to define a topology.

### 1.2 Topology on a Set

A topology $\mathcal{T}$ on a set $X$ is a subset of its power set, $\mathcal{T} \subset P(x)$ satisfying the following three conditions:

- $\emptyset, X \in \mathcal{T}$
- $\forall S \subset \mathcal{T} : (\bigcup_{U \in S} U) \in \mathcal{T}$
- $\forall U_1, \ldots, U_n \in \mathcal{T} : (\bigcap_{i=1}^{i=n} U_i) \in \mathcal{T}$

note that even an infinite union of (intersecting) open intervals of the real line is again an open interval, which agrees with our definition of a topology.

We call a subset $U \subset X$ *open* if $U \in \mathcal{T}$.

### 1.3 Topological Basis

Definition: A subset $\mathcal{B} \subset \mathcal{T}$ is called a *basis* if every element of $\mathcal{T}$ is the union of some subset of $\mathcal{B}$. Symbolically:

$$\forall U \in \mathcal{T}\, \exists C_U \subset \mathcal{B} : U = \bigcup_{B \in C_U} B$$

Given a Basis $\mathcal{B}$ on $X$, a subset $U \subset X$ is open if and only if: $\forall x \in U \exists B \in \mathcal{B} : x \in B \subset U$

### 1.4 Standard Topology on $\mathbb{R}$

We now define the standard topology on $\mathbb{R}$ as the topology with all open intervals as a basis. Notice that in the topology the interval $(a, \infty) := \{x : x > a\}$ is open, because $(x, \infty) = \bigcup_{b>a}(a, b)$, the same holds for $(\infty, a) = \{x : x < a\}$.

## 2 Basic Topological Notions

Having defined a topology, we are now ready to discuss some of the fundamental topics of topology.

### 2.1 Product Topology

Given two sets $A, B$ with respective topologies $\mathcal{T}_A, \mathcal{T}_B$, we define the product topology as:

$$A \times B = \{(a, b) : a \in A, b \in B\} \text{ this is the underlying set}$$

$$\mathcal{T}_{A\times B} \text{ is the topology with basis } \mathcal{B} = \{U \times V : U \in \mathcal{T}_A, V \in \mathcal{T}_B\}$$

Note that $\mathcal{T}_{A\times B} \neq \mathcal{T}_A \times \mathcal{T}_B$. For example two overlapping squares in $\mathbb{R}^2$ is an element of $\mathcal{T}_{\mathbb{R}\times\mathbb{R}}$ but not of $T_{\mathbb{R}} \times T_{\mathbb{R}}$: We can thus define the product topology on $\mathbb{R}^n$ for all natural numbers $n$. We write its basis elements as

$$\prod_i^n U_i = \prod_i^n (a_i, b_i)$$

### 2.2 Comparing Topologies

Given two topologies $\mathcal{T}, \mathcal{T}'$ on the same set $X$, we say that $\mathcal{T}'$ is finer than $\mathcal{T}$ if $\mathcal{T} \subset \mathcal{T}'$. The idea being that in this case everything open in $\mathcal{T}$ is also open in $\mathcal{T}'$, while there are also some more sets open in $\mathcal{T}'$, which gives the topology $\mathcal{T}'$ a more detailed texture.

If a set $X$ has topologies $\mathcal{T}$ and $\mathcal{T}'$ with respective basis $\mathcal{B}$ and $\mathcal{B}'$ such that for each $x \in X$ and every $B \in \mathcal{B}$ with $x \in B$ we have a $B' \in \mathcal{B}'$ satisfying $x \in B' \subset B$ then $\mathcal{T} \subset \mathcal{T}'$. Because let $U \in \mathcal{T}$, then since $U$ is open and $\mathcal{B}$ is a basis for $\mathcal{T}$, for each $x \in U$ we can choose $B_x \in \mathcal{B}$ such that $x \in B_x \subset U$ then for each $B_x$ choose $B'_x \in \mathcal{B}'$ such that $x \in B'_x \subset B_x$ so that we get $U = \bigcup_{x\in U} B'_x$ (since the union of subsets is also a subset, and $\forall x \in U : x \in B'_x$) and thus $U$ is open in $\mathcal{T}'$ and $\mathcal{T} \subset \mathcal{T}'$ as desired.

### 2.3 Closed Sets

We call a set $C \subset X$ *closed* if $C = X - U$ for some open set $U \in \mathcal{T}$ i.e. if its complement $C^c$ is open. Notice that the closed interval $[a, b]$ is in fact a closed set, because $[a, b] = \mathbb{R} - ((\infty, a) \cup (b, \infty))$.

We define the closure $\overline{A}$ of a set $A \subset X$ as the set $\overline{A} = \{x \in X : \forall U \in \mathcal{T} : x \in U \implies U \cap A \neq \emptyset\}$. We will show that (a) $A \subset \overline{A}$, and (b) $\overline{A}$ is closed, so that this is the smallest closed set containing $A$, hence the name.

(a) if $a \in A$ and $a \in U$ then $a \in A \cap U$ so $a \in \overline{A}$ and thus $A \subset \overline{A}$

(b) We have $\forall y \in X - \overline{A}\ \exists U_y \in \mathcal{T} : y \in U_y, U_y \cap A = \emptyset$ but this also gives that $U_y \cap \overline{A} = \emptyset$, for assume by way of contradiction that $z \in U_y \cap \overline{A}$ then $z \in U_y$ and $U_y \cap A = \emptyset$, so in fact $z \notin \overline{A}$. Thus we obtain $X - \overline{A} = \bigcup_{y\notin\overline{A}} U_y$ which is a union of open sets and thus open.

Example: $\overline{(0, 1)} = [0, 1]$ because if $0 \in (a, b)$ then $b > 0$ assume for simplicity that $b < 2$ and so $\frac{b}{2} \in (0, 1) \cap (a, b)$, so that this intersection is nonempty. However, if $x < 0$ then $(2x, \frac{x}{2}) \cap (0, 1) = \emptyset$. A similar argument holds for 1.

There exists sets that are both open and closed. We call such a set clopen. For example in every topology both the empty set and the whole set are clopen. In fact a connected space is exactly a space where only these two are clopen.

# 3 Metric Spaces

We now consider another, more familiar type of topological space. Those which depend on a notion of distance.

## 3.1 Definition

We define a *metric* as a function $d(x, y)$ into $\mathbb{R}$ satisfying these three conditions:

- $d(x, y) = 0$ exactly when $x = y$
- for any $x, y$ we have $d(x, y) = d(y, x)$
- for any $x, y, z$ we have $d(x, z) \leq d(x, y) + d(y, z)$

We can think of a metric as a function which gives the distance between two points. These three conditions state that (1) the only things with no distance between them are a thing and itself, (2) distance doesn't depend on which direction you're going, and (3) distance can't be decreased by stopping at a third point along the way.

**Example 1.** $d(x, y)$ on $\mathbb{R}$ defined by $d(x, y) = |x - y|$

**Example 2.** $d(x, y)$ on $\mathbb{R}^n$ defined by $d(x, y) = |x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$

## 3.2 $\epsilon$-balls

An $\epsilon$-ball $B_\epsilon(x)$ around a point $x$ in a metric space $X$ is the set of all points within $\epsilon$ of $x$ for some real number $\epsilon > 0$. I.e. $B_\epsilon(x) = \{y : d(x, y) < \epsilon\}$.

Thus, a metric $d(x, y)$ induces a topology on a set $X$, which is given by the basis of all $\epsilon$-balls i.e. $\mathcal{B} = \{B_\epsilon(x) : x \in X, \epsilon > 0 \in \mathbb{R}\}$.

**Example 3.** The standard topology on $\mathbb{R}$ is given by the standard metric $d(x, y) = |x - y|$ on $\mathbb{R}$, because $(a, b)$ is the $\epsilon$-ball around $\frac{a+b}{2}$ with radius $\frac{b-a}{2}$.

## 3.3 Equivalent topologies on $\mathbb{R}^n$

We can use our theorem about basis to show that the two topologies on $\mathbb{R}^n$ are equivalent. Given an $\epsilon$-ball $B_\epsilon(x)$ and a point $y \in B_\epsilon(x)$, we can find a basis element of the product topology contained in $B_\epsilon(x)$. Namely we take $\delta = (\epsilon - d(x, y))/n$, and take the basis element $\prod_{i=1}^n (y_i - \delta, y_i + \delta)$. Conversely, gives a basis element $B = \prod_{i=1}^n (a_i, b_i)$ in the product topology, and an element $x \in B$, we can find an $\epsilon$-ball contained in $B$. Namely, we take $\epsilon = Min\{|x_i - a_i|, |x_i - b_i|\}$ i.e. the smallest distance from a coordinate of $x$ to the edge of an interval, and then take $B_\epsilon(x)$. By the way we have chosen $\epsilon$ we must have the desired containment. Since we have containment both ways, we have equality.

# 4 Infinite Dimensional Topologies

We now come to the main definitions which we will be using: the topologies on infinite dimensional real space $\mathbb{R}^\omega$, consisting of all lists of countably many elements of $\mathbb{R}$ e.g. $(\pi, 4.87^3, -e, 101.10010, \ldots)$ and any other such objects.

## 4.1 The Box and Product Topologies

We could define the topology on $\mathbb{R}^\omega$ naively as the topology with basis of all products of open sets in $\mathbb{R}$. Namely $\mathcal{B} = \{\prod_{i \in \mathbb{N}} U_i : U_i \in \mathcal{T}_\mathbb{R}\}$ . This is similar to the product topology in the finite case. We call this the *box topology*. As we will soon see, unlike the finite case, this is not necessarily the most natural topology to put on $\mathbb{R}^\omega$.

Alternatively, we could put a restriction on the above basis, and only include things with finitely many proper subsets of $\mathbb{R}$. So that $U \in \mathcal{B}$ is equal to $\mathbb{R}$ for all but finitely many $U_i$. or symbolically

$$\mathcal{B} = \left\{ \prod_{i=1}^{i=n} U_i \times \prod_{i \in \mathbb{N}} \mathbb{R} : U_i \in \mathcal{T}_{\mathbb{R}} \right\} \text{ here the product is not necessarily ordered as written.}$$

we call this the *product topology* on $\mathbb{R}^\omega$.

## 4.2 The Uniform Topology

We now define the metric topology on $\mathbb{R}^\omega$. The first issue to deal with is the possibility that the distance between two points in $\mathbb{R}^\omega$ might be too large. For example if we consider the distance between $0 = (0, 0, 0, 0, \ldots)$ and $\mathbb{N} = (0, 1, 2, 3, \ldots)$, if we were to use some simple metric this distance would probably be infinite. And so we introduce the notion of a *bounded metric.*

To begin, we define the standard bounded metric on $\mathbb{R}$ as $\overline{d}(x, y) = min\{d(x, y), 1\}$. This is similar to the standard metric, except that it only takes on values in the interval $[0, 1] \subset \mathbb{R}$. This is a metric, since the first two properties follow from the fact that they hold for the standard metric, and we need only be concerned that we might not have $\overline{d}(x, z) \leq \overline{d}(x, y) + \overline{d}(y, z)$ in the case that $2 > d(x, z) > 1$. But even here, the third property of a metric holds rather trivially. Additionally, $\overline{d}$ defines the same metric topology as $d$, as you can check.

Now that we have bounded our metric on $\mathbb{R}$, we are ready to extend it to $\mathbb{R}^\omega$. We define, for a subset $S \subset \mathbb{R}$ the supremum of $S$, denoted $sup(S)$ as the least $x \in \mathbb{R} : \forall y \in S, y \leq x$. So, for two points $x, y \in \mathbb{R}^\omega$ we define the set $S$ of component-wise distances $S = \{\overline{d}(x_i, y_i) : i \in \mathbb{N}\}$. Finally, using this, we can define the *uniform metric* on $\mathbb{R}^\omega$ as $\rho(x, y) = \sup(S) = \sup\{\overline{d}(x_i, y_i) : i \in \mathbb{N}\}$. This gives us a third topology on $\mathbb{R}^\omega$, which we call the *uniform topology.*

## 4.3 Comparing Infinite Topologies

These three topologies relate to each other in precisely the following way:

$$\text{Box} \supsetneq \text{Uniform} \supsetneq \text{Product}$$

**Proof:** (Uniform $\subsetneq$ Box): let $U = \prod_{i \in \mathbb{N}}(\frac{-1}{n}, \frac{1}{n})$ then since $(\frac{-1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ for any $n$, this product is open in the Box topology. However, for any $\epsilon > 0$ there is some $n \in \mathbb{N}$ so that $\frac{1}{n} < \epsilon$ and thus $U$ fails to contain any $\epsilon$-ball. And thus $U$, while open in the box topology, fails to be open in the uniform topology. Thus we have Uniform $\not\supset$ Box. However, given a basis element $B_\epsilon(x)$ in the uniform topology, we can simply choose $U = \prod_{i \in \mathbb{N}}(x_i - \epsilon/2, x_i + \epsilon/2)$ (dividing by 2 is required for the odd edge case where the $x_i$ approach the distance $\epsilon$, so that the supremum ends up bigger than any individual distance, but this point is not particularly central), so that $U$ is contained in $B_\epsilon(x)$. Giving Uniform $\subset$ Box, as desired.

(Product $\subsetneq$ Uniform): for a basis element $B$ in the product topology, there are only finitely many $(a_i, b_i)$ in the product not equal to the whole of $\mathbb{R}$. So we can simply choose $\epsilon = \min\{\frac{b_i - a_i}{2}\}$, giving us the smallest radius of any term in the product giving $B$, then we have $B_\epsilon(x) \subset B$ where $x_i = \frac{a_i + b_i}{2}$ for $U_i \neq \mathbb{R}$ and $x_i = 0$ otherwise. So this gives us Product $\subset$ Uniform. Conversely, consider $B_1(0)$, which is a basis element in the uniform topology. There is clearly no basis element in the product topology contained in it, this gives us Product $\not\supset$ Uniform.

# 5 Closure of Sequences that are Eventually Zero

We now look at a special subset of $\mathbb{R}^\omega$, and consider its closure in the above topologies. We define $\mathbb{R}^\infty \subset \mathbb{R}^\omega$ as the set of elements in $\mathbb{R}^\omega$ with only finitely many nonzero entries. We consider $\overline{\mathbb{R}^\infty}$ in the box, product and uniform topologies.

### 5.1 The Box Topology

As per usual, we trivially have that $\mathbb{R}^\infty \subset \overline{\mathbb{R}^\infty}$. We show that this is in fact all of $\overline{\mathbb{R}^\infty}$, or rather that $\mathbb{R}^\infty$ is already closed in the box topology. Choose any $y \in \mathbb{R}^\omega$ such that $y \notin \mathbb{R}^\infty$, then we can choose $U = \prod(y_i - |\frac{y_i}{2}|, y_i + |\frac{y_i}{2}|)$ so that $0 \notin U_i$ and thus $U \cap \mathbb{R}^\infty = \emptyset$ so that $y \notin \overline{\mathbb{R}^\infty}$ and $\mathbb{R}^\infty = \overline{\mathbb{R}^\infty}$.

### 5.2 The Product Topology

Let $y \in \mathbb{R}^\omega$ be anything, then $y \in \overline{\mathbb{R}^\infty}$. For, consider any open set $U$ containing $y$, then $U_i = \mathbb{R}$ for all but finitely many $i$, so that $0 \notin U_i$ for at most finitely many $i$. Let $x \in \mathbb{R}^\infty$ be defined as $x_i = y_i$ when $U_i \neq \mathbb{R}$ and $x_i = 0$ whenever $U_i = \mathbb{R}$. We thus have $x \in U \cap \mathbb{R}^\infty$ so that $y \in \overline{\mathbb{R}^\infty}$ for all $y \in \mathbb{R}^\omega$, and $\overline{\mathbb{R}^\infty} = \mathbb{R}^\omega$, as desired.

### 5.3 The Uniform Topology

In the uniform topology, the closure of sequences which are eventually zero is exactly sequences that converge to zero. Let $y \in \mathbb{R}^\omega$ be such that as $i \to \infty$ we get $y_i \to 0$ (for example $y_n = 1/n$), and let $U$ be an open set in the uniform topology containing $y$. By definition of basis, there is some $B \in \mathcal{B} : y \in B \subset U$. But these basis elements are of the form $B_\epsilon(y)$. Then by the definition on convergence we have $\forall \epsilon > 0 \exists n \in \mathbb{N} : i > n \implies |y_i| < \epsilon$. In particular, for some $n \in \mathbb{N}$ for every $i > n$ we get $\overline{d}_i(y_i, 0) < \epsilon$. So define $x$ as $x_i = y_i$ for $i \leq n$ and $x_i = 0$ for $i > n$ then $x \in B_\epsilon(y) \cap \mathbb{R}^\infty$ but since $B \subset U$ we also have $x \in U \cap \mathbb{R}^\infty$ so that $y \in \overline{\mathbb{R}^\infty}$.

Conversely, suppose that $y \in \mathbb{R}^\infty$ does not converge to 0. This means, by the contrapositive of the above, that $\exists \epsilon > 0 \forall n \in \mathbb{N} : \exists i > n : y_i > \epsilon$. Taking such epsilon, we choose the neighborhood around $y$ given by $B = B_{\frac{\epsilon}{2}}(y)$ so that there are infinitely many $B_i$ with $0 \notin B_i$ and thus $B \cap \mathbb{R}^\infty = \emptyset$ and $y \notin \overline{\mathbb{R}^\infty}$.

### 5.4 Conclusion

Although it is essentially possible to imagine that $\mathbb{R}^\infty$ is already closed, or that its closure is everything in $\mathbb{R}^\omega$. Since the three topologies on $\mathbb{R}^\omega$ all agree on $\mathbb{R}^n$, we are in some sense free to choose which topology we'd like to put on $\mathbb{R}^\omega$. And it can be seen as most sensible that the closure of $\mathbb{R}^\infty$, i.e. the things that are very close to it, should be exactly the things that approach $\mathbb{R}^\infty$ indefinitely.